XLVIII. *Concise Rules for computing the Effects of Refraction and Parallax in varying the Apparent Distance of the* Moon *from the* Sun *or a* Star; *also an easy Rule of Approximation for computing the Distance of the* Moon *from a* Star, *the Longitudes and Latitudes of both being given, with Demonstrations of the same: By the Rev.* Nevil Maskelyne, *A. M. Fellow of* Trinity College, *in the University of* Cambridge, *and F. R. S.*

Read Nov. 15, 1764.

THE following rules, excepting one, are the same which I have already communicated to the Royal Society, but without demonstration, in a letter to the reverend Dr. Birch from St. Helena, containing the results of my observations of the distance of the Moon from the Sun and fixed stars, taken in my voyage thither, for finding the longitude of the ship from time to time; since printed in Part II. Vol. LII. of the Philosophical Transactions for 1762. The two rules for the correction of refraction and parallax I have also since communicated to the public in my British Mariner's Guide to the discovery of longitude from like observations of the Moon; and have added in the Preface a rule for computing a second but smaller correction of parrallax, necessary on account of a small imperfection

imperfection lying in the first rule derived from the fluxions of a spherical triangle. To the rules I have here subjoined their demonstrations.

With respect to the usefulness of these rules, I cannot but entertain hopes that they will appear more simple and easy than any yet proposed for the same purpose: the last rule, for computing the distance of the Moon from a star, though only an approximation, being so very exact, seems particularly adapted for the construction of a nautical Ephemeris, containing the distances of the Moon from the Sun and proper fixed stars ready calculated for the purpose of finding the longitude from observations of the Moon at sea; an assistance which, in an age abounding with so many able computers, mariners need not doubt they will be provided with, as soon as they manifest a proper disposition to make use of it.

A RULE

To compute the contraction of the apparent distance of any two heavenly bodies by refraction; the zenith distances of both, and their distance from each other being given nearly.

Add together the tangents of half the sum, and half the difference of the zenith distances; their sum, abating 10 from the index, is the tangent of arc the first. To the tangent of arc the first, just found, add the co-tangent of half the distance of the stars; the sum, abating 10 from the index, is the tangent of arc the second. Then add together the tangent of double the first arc, the co-secant of double the second

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Fig. 1.

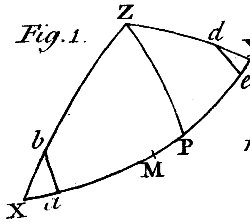


Fig. 2.

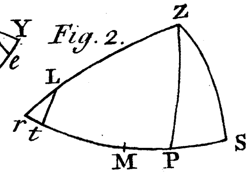


Fig. 3.

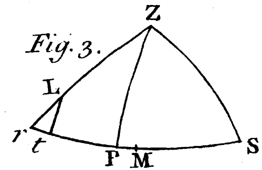


Fig. 4.

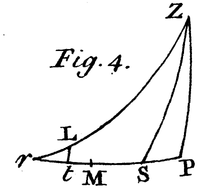


Fig. 5.

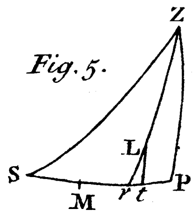


Fig. 6.

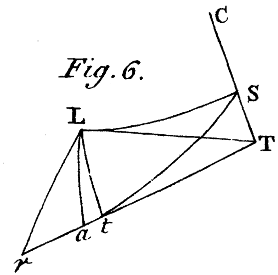
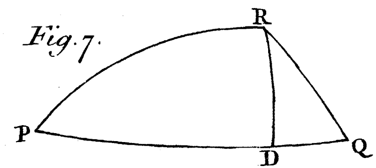


Fig. 7.



J. Mynde sc.

double the fecond arch, and the conftant logarithm of 114″ or 2,0569: the fum abating 20 from the index, is the logarithm of the number of feconds required, by which the diftance of the ftars is contracted by refraction: which therefore added to the obferved diftance gives the true diftance cleared from the effect of refraction.

Explication of the foundation of the preceding rule.

This rule is founded upon an hypothefis that the refraction in altitude is as the tangent of the zenith diftance: and the refraction at the altitude of 45 degrees being 57″, according to Dr. Bradley's obfervations, therefore the refraction at any altitude, calling the radius unity is $= 57'' \times$ tangent of the zenith diftance. This rule is exact enough for the purpofe of the calculation of the longitude from obfervations of the diftance of the Moon from ftars at fea as low down as the altitude of 10°, for there the error is only 10″ from the truth. But, if the altitude of the Moon or ftar be lefs than 10°, the rule may be ftill made to anfwer fufficiently, by only firft correcting the obferved zenith diftances by fubftracting from them three times the refraction correfponding to them taken out of any common table of refraction, and making the computation with the zenith diftances thus corrected. This correction depends upon Dr. Bradley's rule for refraction, which he found to anfwer, in a manner exactly, from the zenith quite down to the horifon, namely that the refraction is $= 57'' \times$ tangent of the apparent zenith diftance

 leffened

leſſened by three times the correſponding refraction taken out of any common table.

Demonſtration of the preceding rule.

Let ZXY Tab. XVII. Fig. 1. repreſent a ſpherical triangle formed by great circles joining the zenith Z and the ſtars X and Y. Refraction acting in the vertical circles ZX and ZY will carry the ſtar X nearer the zenith by a quantity $Xb = 57'' \times$ tangent of ZX, and the ſtars Y towards Z by the quantity $Yd = 57'' \times$ tangent of ZY; ſo that the apparent diſtance of the two ſtars will be bd inſtead of XY or leſs than XY, the true diſtance, by the Sun of the two little ſpaces Xa, Ye, terminated by the perpenndiculars ba and de. The little ſpace $Xa = Xb \times$ coſine of the angle ZXY (calling radius unity) $= 57'' \times$ tang. of $ZX \times$ coſine of angle ZXY, or, by ſpherics, $= 57'' \times$ tang. of XP. (ZP being an arch drawn from Z perpendicular to the arch XY). In like manner the little ſpace $Ye = 57'' \times$ tang. of YP; and therefore $Xa + Ye$ or the total effect of refraction $= 57'' \times \overline{\text{tang. XP} + \text{tang YP}}$. Let M be the middle of the arch XY, and put the tangent of XM or YM or $\frac{1}{2}XY = t$, and the tangent of MP, or the diſtance of the perpendicular ZP from M the middle of the arch $XY = n$. By trigonometry, tang. of XP or $XM + MP = \frac{t+n}{1-tn}$, and tang. of YP or $YM - MP = \frac{t-n}{1+tn}$; the ſum of which, or tang. $XP +$ tang. $YP = \frac{t+n}{1-tn} + \frac{t-n}{1+tn} = \frac{2t+2tn^2}{1-t^2n^2} = \frac{2tn}{1-t^2n^2} \times \frac{1+n^2}{2n} \times 2$. Now

$\frac{2tn}{1-t^2n^2}$ is the tangent of double the angle whoſe tangent is tn, and tn or the product of the tangents of $\frac{1}{2}$ XY and MP, by ſpherics, is equal to the product of the tangents of half the ſum and half the difference of the zenith diſtances ZX and ZY; whence $\frac{2tn}{1-t^2n^2}$ is equal to the tangent of double arch the firſt found by the rule. Alſo arch the ſecond found by the rule being by ſpherics = MP, whoſe tangent is repreſented by n, and $\frac{1+n^2}{2n}$ being by trigonometry equal to the coſecant of double the arch whoſe tangent is n, therefore $\frac{1+n^2}{2n}$ = coſecant of 2 MP or double arch the ſecond. Whence the rule is manifeſt; namely, that Xa + Ye, the total effect of refraction in contracting the apparent diſtance of the two ſtars $= 57'' \times \overline{\text{tan. XP} + \text{tan. YP}}$ = double 57″ or 114″ × tang. of double the firſt arch × coſecant of double the ſecond arch. Q. E. D.

R E M A R K.

When the perpendicular arch ZP, let fall from the Zenith on the arch XY, falls without the triangle ZXY, the effect of refraction in diminiſhing the apparent diſtance of the ſtars X, Y is the difference of Xa and Ye: but the rule being general, gives always the ſum or difference, which-ever it be, which is a great advantage, and removes all grounds of ambiguity in the correction of refraction; as the the total effect thereof always diminiſhes the diſtance of two ſtars from each other, however they are poſited.

A RULE

To compute the contraction or augmentation of the apparent diſtance of the Moon from a ſtar, on account of the Moon's parallax; the zenith diſtances of the Moon and ſtar, and their diſtance from each other being given nearly.

Add together the tangents of half the ſum, and half the difference of the zenith diſtances of the Moon and ſtar, and the cotangent of half the diſtance of the Moon from the ſtar; the ſum, abating 20 from the index, is the tangent of an arch, which call A. Then, if the zenith diſtance of the Moon is greater than that of the ſtar, take the Sum of the arch A, juſt found, and half the diſtance of the Moon from the ſtar; but, if the zenith diſtance of the Moon be leſs than that of the ſtar, take the difference of the ſaid arch A and half the diſtance of the Moon from the ſtar; and the ſum, or difference call B. To the tangent of B, thus found, add the coſine of the Moon's zenith diſtance, and the logarithm of the Moon's horizontal parallax, expreſſed in minutes and decimals; the ſum, abating 20 from the index, is the logarithm of the effect of parallax, tending always to augment the apparent diſtance of the Moon from the ſtar; except the zenith diſtance of the Moon be leſs than that of the ſtar, and, at the ſame time, the arch A be greater than half the diſtance of the Moon from the ſtar, in which caſe the effect of parallax diminiſhes the apparent diſtance of the Moon from the ſtar.

DEMONSTRATION.

In the ſpherical triangle ZLS, ſee Fig. 2, 3, 4, and 5th, Z repreſents the zenith, L the Moon, and S the ſtar; the effect of parallax depreſſing the Moon from L to r, r is the apparent place of the Moon, and rS the apparent diſtance of the Moon from the ſtar; let fall the perpendicular Lt upon rS, produced if neceſſary, and rt will be the difference of LS and rS, or the effect of parallax. Draw the arch ZP perpendicular to to rS, and let M be the middle of rS. The Moon's parallax in altitude, being to her horizontal parallax, as the ſine of her apparent zenith diſtance, to the radius, Lr = Moon's horizontal parallax × ſine of Zr; and rt the effect of parallax upon the apparent diſtance of the Moon from the ſtar will be = Lr × coſ. ZrS = horizontal parallax × ſin. Zr × coſ. ZrS (or, becauſe tan. rP : coſ. ZrP :: tan Zr : rad :: ſin Zr : cos Zr; and therefore ſin. Zr × cos ZrS = coſ. Zr × tan rP) = horizontal parallax × coſ. Zr × tan. rP agreeably to the rule. For it is evident by ſpherics that the arch A, found by the rule, is the ſame with MP the diſtance of the perpendicular from the middle of the arch rS: and it is evident, by the inſpection of the figures, that the arch B or rP is equal to the ſum of rM and MP, if the zenith diſtance of the Moon be greater than that of the ſtar, as in Fig. 2d and 4th; but is the difference of rM and MP, if the zenith diſtance of the Moon be leſs than that of the ſtar, as in Fig. 3d and 5th. Laſtly, it may appear from the conſideration

consideration of the figures, that, as the effect of parallax depresses the Moon directly towards the horizon, so it will always encrease her apparent distance from a star, except in the case represented by Fig. 5th; that is to say, unless the zenith distance of the Moon be less than that of the star, and, at the same time, the arch MP be greater than rM or half the distance of the Moon from the star. Q. E. D.

Remarks on the use of the two foregoing rules.

It has been remarked, after the rule for refraction above, that if the altitudes of the Moon or star are under 10 degrees, the zenith distances must be first lessened by 3 times the refractions corresponding to their respective altitudes before the effect of refraction be computed.

But in order to compute the effect of parallax from the second rule, the observed distance of the Moon from the star must be first corrected by adding the effect of refraction to it found by rule the first; as must the observed altitudes of the Moon and star be also corrected by taking from them their respective refraction in altitude, and the corrected arches thus found must be made use of in computing the parallax. Only, if the altitudes of the Moon and star are both 10 degrees or more, part of the calculation of rule the second may be saved, and arch the second, found by rule the first, taken for arch A in the second rule without any sensible error. In this case, it will be most convenient to observe the following order of computation instead of that before prefcribed to be ufed when the altitudes are under 10 degrees.

1ft. Making ufe of the apparent altitudes of the Moon and ftar uncorrected, compute arches the firft and fecond by the directions contained in the rule of refraction.

2dly. Taking arch the fecond for arch A in the rule of parallax, compute the effect of parallax according to rule the fecond.

3dly. With arches the firft and fecond compute the effect of refraction by rule the firft.

4thly, and laftly. Applying the two corrections of parallax and refraction duly, according to the rules, to the obferved diftance of the Moon from the ftar, you will have the true and correct diftance of the Moon from the ftar, cleared both of refraction and parallax.

A RULE

For computing a fecond, but fmaller correction than the firft, neceffary to be applied to the obfervations of the diftance of the Moon from a ftar on account of parallax.

Call the principal effect of parallax, found by the preceding rule, the parallax in diftance; and find the parallax anfwering to the Moon's altitude. Then to the conftant logarithm 0.941 add the logarithm of the fum of the parallax in altitude and the parallax in diftance, the logarithm of the difference of the fame parallaxes, and the cotangent of the obferved diftance of the Moon from the ftar (corrected for refraction, and the principal effect of parallax), the fum, abating 13 from the index, is the logarithm

logarithm of the number of ſeconds required, being the ſecond correction of parallax; and is always to be added to the diſtance of the Moon from the ſtar, firſt corrected for refraction, and the principal effect of parallax found above, in order to obtain the true diſtance; unleſs the diſtance exceeds 90 degrees, in which caſe it is to be ſubſtracted.

DEMONSTRATION.

Let L Fig. 6. repreſent the Moon's true place in the ſphere, and *r* her apparent place as depreſſed by parallax, S the place of the ſtar, and L*t* a perpendicular let fall from the true place of the Moon L upon the great circle *r*S joining the ſtar S and the apparent place of the Moon *r*; (all as in the four figures belonging to the preceding rule). Let L*a* be the arch of a parallel circle deſcribed from the ſtar S as a pole through the true place of the Moon L. S*a* terminated by the parallel circle L*a*, and not S*t* terminated by the perpendicular L*t*, as was ſuppoſed in the former demonſtration, is equal to SL or the true diſtance of the Moon from the ſtar, which was therefore computed too ſmall from the former rule by the little ſpace *at*. Let LT and *a*T be the equal tangents of the equal arches LS and *a*S in L and *a*, meeting in the radius CS, drawn from the centre of the ſphere C and produced, in T. The ſpace L*at*, on account of its ſmallneſs, may be looked upon as lying all in one plane namely L*a*T, and L*a* as the ſmall arch of a circle deſcribed from the point T as a centre with the line LT as a radius, thro' through L and a, Lt as the ſine, and at as the verſe-ſine of the arch La; and conſequently at equal to the ſquare of Lt divided by 2 L T. But, the triangle Lrt being right-angled in t, the ſquare of Lt is equal to the difference of the ſquares of Lr and rt, and conſequently to the product of their ſum and difference; that is to ſay, $at = \frac{\overline{Lr+rt} \times \overline{Lr-rt}}{2\,LT}$ or (becauſe the tangent T L is equal to the ſquare of the radius C S divided by the cotangent of L S) $= \overline{Lr+rt} \times \overline{Lr-rt} \frac{\times \text{cotangent of LS}}{2 \times \text{ſquare of CS}}$. Now ſuppoſe the ſpaces Lr, rt to be expreſſed in minutes, which will be moſt convenient in practice, then the radius of the ſphere C S muſt be taken equal to $3437\frac{3}{4}$, for ſo many minutes are contained in an arch of a circle equal to its radius: and at will be $=$ $\frac{\overline{Lr+rt} + \overline{Lr-rt} \times \text{cotan. of LS}}{2 \times 3437\frac{3}{4} \times 3437\frac{3}{4}}$. But, the contangents of ſimilar arches of circles of different radii being directly as the radii, therefore the cotangent of L S to the radius C S or $3437\frac{3}{4}$, is to the cotangent of the ſame arch to 10000000000, which is the radius to which the logarithmic tables are adapted, its logarithm being 10; as $3437\frac{3}{4}$ to 10000000000. Therefore the cotangent of L S $=$ tabular cotangent $\frac{\text{of LS} \times 3437\frac{3}{4}}{10000000000}$, which, being ſubſtituted in the value of at above, gives at, expreſſed in minutes $=$ $\frac{\overline{Lr+rt} + \overline{Lr-rt} \times \text{tabular cotangent LS}}{6875500000000}$; or, multiplying by 60, the value of at will come out in ſe-

 conds

conds $= \frac{\overline{Lr + rt} \times \overline{Lr - rt} \times \text{tabular cotangent of L S}}{1146000000000}$.

The logarithm of the denominator 1146000000000 is 12,059, instead of subtracting which, when the operation is performed by logarithms, add 0,941 (its compliment to 13) to the value of the numerator found in logarithms, and substract 13 from the index: the remainder will be the value of *at* in seconds. Q. E. D.

A concise rule to find the distance of the Moon from a zodiacal star, very nearly; the difference of the longitudes of the Moon and star, and the latitudes of both being given.

To the cosine of the difference of the longitudes add the cosine of the difference of the latitudes, if both of the same denomination, or sum; if of contrary denominations, the sum of the two logarithms, abating 10 from the index, is the cosine of the approximate distance. This gives the true distance of the Moon from the Sun, being then nothing more than the common rule for finding the hypothenuse of a right-angled spherical triangle from the two sides given. But in the case of a zodiacal star apply the following correction to the approximate distance thus found.

To the constant logarithm 5.3144 add the sine of the Moon's latitude, the sine of the star's latitude, the verse-sine of the difference of longitude, and the cosecant of the approximate distance; the sum of these 5 logarithms, abating 40 from the index, is the logarithm of a number of seconds, which substracted

ſtracted from the approximate diſtance, found before, if the latitudes of the moon and ſtar are of the ſame denomination, or added thereto, if they are of different denominations, gives the true diſtance of the Moon from the ſtar.

N. B. This rule, though only an approximation, is ſo very exact, that even, if the latitude of the Moon was 5°, and that of the ſtar 15°, the error would be only 10″; and if the latitude of the Moon be 5°, and that of the ſtar 10°, the error is only 4″ $\frac{1}{2}$; and if the latitudes be leſs, will be leſs in proportion as the ſquares of the ſines of the latitudes decreaſe.

DEMONSTRATION.

Let P [Fig. 7.] repreſent one of the poles of the eclip- and Q, R the places of the Moon and ſtar. From R let the arch of a great circle R D be drawn perpendicular to P Q. By ſpherics, the tangent of P D = tangent of P R × coſine of the angle R P D. And, by trigonometry, coſine of Q D or (Q P — P D) = coſ. Q P × coſ. P D + ſin. Q P × ſin. P D = coſ. Q P × coſ. P D + ſin. Q P × coſ. P D × tan. P D = coſ. P D × $\overline{\text{coſ. QP + ſin. QP × tan. PD}}$ = coſ. P D × $\overline{\text{coſ. QP + ſin. QP × tan. PR × coſ. P}}$ ∴ coſ. Q D : coſ. P D :: coſ. Q P + ſin. Q P × tan. P R × coſ. P : 1. But, by ſpherics, coſ. Q D : coſ. P D :: coſ. R Q : coſ. P R ∴ coſ. R Q : coſ. P R :: coſ. Q P + ſin. Q P × tan. P R × coſ. P : 1. Whence coſ. R Q = coſ. P R × coſ. Q P + ſin. Q P × ſin. P R × coſ. P. Now, by trigonometry, coſ. (Q P — P R)

$=$ cos. Q P $\times$ cos. P R $+$ sin. Q P $\times$ sin. P R; whence sin. Q P $\times$ sin. P R $=$ cos. (Q P $-$ P R) $-$ cos. Q P $\times$ cos. P R; which being substituted above, gives cos. R Q $=$ cos. (Q P $-$ P R) $\times$ cos. P $-$ cos. P R $\times$ cos. Q P $\times$ cos. P $+$ cos. P R $\times$ cos. Q P $=$ cos. (Q P $-$ P R) $\times$ cos. P $+$ verse-sine P $\times$ cos. P R $\times$ cos. Q P. Now put cos. (Q P $-$ P R) $\times$ cos. P $=$ cos. G, or the approximate distance, then cos. R Q $-$ cos. G, or (because the difference of R Q and G is but small) $\overline{G - RQ} \times$ sin. $\left(\frac{G + RQ}{2}\right)$ $=$ verse-sine P $\times$ cos. P R $\times$ cos. P Q nearly. Whence

$$RQ = G - \frac{\text{verse-sine } P \times \text{cos. } PR \times \text{cos. } QP}{\text{sin. } \frac{G + RQ}{2}}$$

nearly

$$= G - \frac{\text{v. s. } P \times \text{cos. } PR \times \text{cos. } QP}{\text{sin. } G}$$

nearly. Q. E. D.

Note, the error of this formula arises from taking $G = \frac{G + RQ}{2}$ by which means it will always give R Q too great, nearly by the following quantity, $\frac{1}{2}$ S q cos. P R $\times$ S q cos. Q P $\times$ cot. G $\times$ S q tan. $\frac{1}{2}$ G. This comes to a maximum when G is 60°, and is then $= \frac{1}{6} \sqrt{\frac{1}{3}} \times$ S q cos. R P $\times$ S q cos. P Q. If the latitudes of the Moon and star are both 5° it is $= 1''$. If the Moon's latitude be 5°, and that of the star 10°, it is $= 4'' \frac{1}{2}$; and if the latitude of the star be 15° it is $= 10''$.

XLIX.